Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Journal of Geometry and Physics 56 (2006) 1728–1735

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

On the mean curvature of spacelike submanifolds in semi-Riemannian manifolds

BingYe Wu

Department of Mathematics, Minjiang University, Fuzhou, Fujian 350108, China

Received 2 June 2005; received in revised form 4 October 2005; accepted 5 October 2005
Available online 21 November 2005

Abstract

In this paper we establish some estimates for the higher-order mean curvature of a complete spacelike hypersurface in spacetimes with sectional curvature satisfying certain condition. We also obtain the estimate for the mean curvature of a complete spacelike submanifold in semi-Riemannian space forms.

MSC 2000: Primary 53C42, Secondary 53B30

Keywords: Spacetime; Semi-Riemannian space form; Lorentzian distance; Mean curvature

1. Introduction

Spacelike submanifolds in semi-Riemannian manifolds have been of increasing interest in the recent years from various points of view, especially in the case of hypersurfaces. For instance, In reference [1] Aledo and Alías characterized the spacelike hyperplanes as the only complete spacelike hypersurfaces with constant mean curvature in Lorentz–Minkowski space which lie in two parallel spacelike hyperplanes, and the hyperbolic spaces as the only complete spacelike hypersurfaces with constant mean curvature in Lorentz–Minkowski space which lie in two concentric hyperbolic spaces. They also obtained some estimates for the higher-order mean curvatures of complete spacelike hypersurfaces in Lorentz–Minkowski space which are bounded by hyperbolic spaces. Similar estimates also holds for spacelike hypersurfaces in de Sitter space [2].

Note that both the Lorentz–Minkowski space and the de Sitter space belong to the classical Robertson-Walker spacetimes which are spatially homogeneous. Since the spatial homogeneity

E-mail address: bingyewu@yahoo.com.cn.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.10.008

could not be realistic, the classical Robertson-Walker spacetimes are only appropriate as a rough approach to consider the universe in the large, but not to consider it in a more accurate scale. Thus it is natural to study spacelike hypersurfaces in more general spacetimes, and that has been considered for generalized Robertson-Walker spacetimes (see e.g.,[3]). In this paper, we shall consider spacelike hypersurface in more general spacetimes which lies in a certain open set in the chronological future or chronological past of a point such that, the corresponding Lorentzian distance function is differentiable on it. Physically, the events we may experience in the universe are the ones in our chronological future. One of the tools to analyze the geometry of our chronological future is the Lorentzian distance function, and we shall use it together with the generalized maximum principle [7] to establish some estimates for the higher order mean curvatures of complete spacelike hypersurfaces in general spacetimes with sectional curvature satisfying certain condition. We also obtain the estimate for the mean curvature of complete spacelike submanifolds in semi-Riemannian space forms. In order to state our results, let us first recall the relative result in [1] and fix some notations.

Let $(R_1^{n+1}, \langle , \rangle)$ be the Lorentz–Minkowski $(n+1)$-space. First we have

Proposition 1. ([1]) *Let $\psi : M^n \to R_1^{n+1}$ be a connected and complete spacelike hypersurface in Lorentz–Minkowski space whose sectional curvatures are bounded away from $-\infty$. If*

$$\sup_M \langle \psi - a, \psi - a \rangle = -r^2 \tag{1}$$

for some $a \in R_1^{n+1}$ and $r > 0$, then for $j = 1, \ldots, n$,

$$\sup_M \mid H_j \mid \geq \frac{1}{r^j},$$

where H_j is the jth mean curvature of ψ. Consequently, there exists no complete spacelike hypersurface $\psi : M^n \to R_1^{n+1}$ whose sectional curvatures are bounded away from $-\infty$ and that $H_j = 0$ for some odd j or $H_j \leq 0$ for some even j and $\psi(M) \subset \Omega(a, r)$ for some $a \in R_1^{n+1}$ and $r > 0$, where

$$\Omega(a, r) = \{x \in R_1^{n+1} : \langle x - a, x - a \rangle \leq -r^2\} \subset R_1^{n+1}. \tag{2}$$

Our first result is to generalize Proposition 1 into the case where R_1^{n+1} is replaced by general spacetime whose sectional curvature satisfying certain condition. For this purpose, let us interpret condition (1) in term of Lorentzian distance d on R_1^{n+1}. Let $I^+(a)$ and $I^-(a)$ be the chronological future and chronological past of a in R_1^{n+1}, respectively. Then by the definition of Lorentzian distance (see e.g., [6] or [4]) it is easy to know that the condition (1) is equivalent to

$$\inf_M d(a, \psi) = r$$

if $\psi(M) \subset I^+(a)$ or

$$\inf_M d(\psi, a) = r$$

if $\psi(M) \subset I^-(a)$, while the definition (2) of $\Omega(a, r)$ can be rewritten as

$$\Omega(a, r) = \{x \in R_1^{n+1} : d(a, x) \geq r \quad \text{or} \quad d(x, a) \geq r\}.$$

Let N_1^{n+1} be a spacetime of signature $(1, n)$ and $d : M \times M \to R \cup \{\infty\}$ be the corresponding Lorentzian distance function. We note that for general spacetime the Lorentzian distance function with respect to a given point is not differentiable on the chronological future or chronological past of the point. In fact it is not even continuous in general on a spacetime other than a globally hyperbolic spacetime. Nevertheless, in strongly causal spacetimes, Lorentzian distance function with respect to a point is differentiable at least in a sufficiently near chronological future or chronological past of the point. For $a \in N_1^{n+1}$ let $\mathcal{I}^+(a)$ (resp. $\mathcal{I}^-(a)$) be the open set such that $d(a, \cdot)$ (resp. $d(\cdot, a)$) is differentiable on it. We shall give the precise definitions of $\mathcal{I}^+(a)$ and $\mathcal{I}^-(a)$ in Section 2. For $c \in R$, $r > 0$, let

$$\alpha(c,r) = \begin{cases} \sqrt{c}\coth(\sqrt{c}r), & c > 0; \\ \frac{1}{r}, & c = 0; \\ \sqrt{-c}\cot(\sqrt{-c}r), & c < 0. \end{cases}$$

We shall prove the following.

Theorem 2. *Let N_1^{n+1} be a spacetime of signature $(1, n)$ with $K(\pi) \geq c$ for any mixed 2-plane π of signature $(1, 1)$ in N_1^{n+1}, where $K(\cdot)$ denotes the sectional curvature. Assume that for $a \in N_1^{n+1}, \mathcal{I}^+(a) \neq \emptyset$ (resp. $\mathcal{I}^-(a) \neq \emptyset$). Let $\psi : M^n \to N_1^{n+1}$ be a complete spacelike hypersurface in N_1^{n+1} whose sectional curvatures are bounded away from $-\infty$. If $\psi(M) \subset \mathcal{I}^+(a)$ (resp. $\psi(M) \subset \mathcal{I}^-(a)$) and*

$$\inf_M d(a,\psi) = r \quad (\text{resp.} \inf_M d(\psi,a) = r)$$

for some $r > 0$ (when $c < 0$, we assume that $r < \pi/(2\sqrt{-c})$), then for $j = 1, \cdots, n$,

$$\sup_M | H_j | \geq \alpha(c,r)^j.$$

Consequently, there exists no complete spacelike hypersurface $\psi : M^n \to N_1^{n+1}$ whose sectional curvatures are bounded away from $-\infty$, and that $H_j = 0$ for some odd j or $H_j \leq 0$ for some even j and $\psi(M) \subset \Omega(a, r)$ for some $a \in N_1^{n+1}$ and $r > 0$ (when $c < 0$, we assume that $r < \pi/(2\sqrt{-c})$), where

$$\Omega(a,r) = \{x \in N_1^{n+1} : d(a,x) \geq r\} \cap \mathcal{I}^+(a).$$

If N_1^{n+1} has nonnegative timelike sectional curvature, it is easy to know from Proposition 11.15 of [4] that N_1^{n+1} has no future and past timelike conjugate points. With this in mind, we know from the Lorentzian Hadamard-Cartan theorem (see [4], p. 414) that if N_1^{n+1} is a future one-connected globally hyperbolic spacetime with nonnegative timelike sectional curvature, then the Lorentzian distance function with respect to a given point is differentiable on the chronological future or chronological past of the point, i.e., $\mathcal{I}^+(a) = I^+(a)$ and $\mathcal{I}^-(a) = I^-(a)$. We recall that a spacetime is said to be future one-connected if any two smooth, future directed timelike curves from a to b are homotopic through (smooth) future directed timelike curves with fixed endpoints a and b. Therefore, by Theorem 2 we have the following result which recover Proposition 1.

Corollary 3. *Let N_1^{n+1} be a future one-connected globally hyperbolic spacetime of signature $(1, n)$ with $K(\pi) \geq c \geq 0$ for any mixed 2-plane π of signature $(1, 1)$ in N_1^{n+1}, and $\psi : M^n \to N_1^{n+1}$ be a complete spacelike hypersurface in N_1^{n+1} whose sectional curvatures are bounded away from $-\infty$. If*

$$\inf_M d(a,\psi) = r \quad \text{or} \quad \inf_M d(\psi,a) = r$$

for some $a \in N_1^{n+1}$ *and* $r > 0$, *then for* $j = 1, \cdots, n$,

$$\sup_M \mid H_j \mid \geq \alpha(c, r)^j.$$

Consequently, there exists no complete spacelike hypersurface $\psi : M^n \to N_1^{n+1}$ *whose sectional curvatures are bounded away from* $-\infty$ *and that* $H_j = 0$ *for some odd* j *or* $H_j \leq 0$ *for some even* j *and* $\psi(M) \subset \Omega(a, r)$ *for some* $a \in N_1^{n+1}$ *and* $r > 0$, *where*

$$\Omega(a, r) = \{x \in N_1^{n+1} : d(a, x) \geq r \quad \text{or} \quad d(x, a) \geq r\}.$$

Let $N_p^{n+p}(c)$ be the simply connected semi-Riemannian space form of signature (p, n) with constant sectional curvature c, i.e.,

$$N_p^{n+p}(c) = \begin{cases} S_p^{n+p}(c) \subset R_p^{n+p+1}, & c > 0; \\ R_p^{n+p}, & c = 0; \\ H_p^{n+p}(c) \subset R_{p+1}^{n+p+1}, & c < 0. \end{cases}$$

Let $\langle , \rangle$ be the inner product on corresponding semi-Euclidean space. Our second main result is the following.

Theorem 4. *Let* $N_p^{n+p}(c)$ *be the simply connected semi-Riemannian space form of signature* (p, n) *with constant sectional curvature* c *and* $\psi : M^n \to N_p^{n+p}(c)$ *be a complete spacelike submanifold of signature* n *in* $N_p^{n+p}(c)$ *whose Ricci curvatures are bounded away from* $-\infty$. *If*

$$\inf_M \langle a, \psi \rangle = \frac{1}{c} \cosh(\sqrt{c}r)$$

when $c > 0$ *or*

$$\sup_M \langle \psi - a, \psi - a \rangle = -r^2$$

when $c = 0$ *or*

$$\sup_M \mid \langle a, \psi \rangle \mid = \frac{1}{-c} \cos(\sqrt{-c}r)$$

when $c < 0$, *for some* $a \in N_p^{n+p}(c)$ *and* $r > 0$ (*when* $c < 0$, *we assume that* $r < \pi/(2\sqrt{-c})$). *Then* $\sup_M \mid \mathbf{H} \mid \geq \alpha(c, r)$, *where* $\mathbf{H}$ *is the mean curvature vector of* ψ.

2. Preliminaries

Let $(N_1^{n+1}, \langle\cdot\rangle)$ be a spacetime of signature $(1, n)$, and $d : M \times M \to R \cup \{\infty\}$ be the corresponding Lorentzian distance function. Let $T_{-1}N \mid_a = \{v \in T_a N : v \text{ is future directed and } \langle v, v \rangle = -1\}$ be the fiber of the unit future observer bundle $T_{-1}N$ at a. Define the function $s_a : T_{-1}N \mid_a \to R \cup \{\infty\}$ by $s_a(v) = \sup\{t \geq 0 : d(a, \gamma_v(t)) = t\}$, where $\gamma_v : [0, r) \to N_1^{n+1}$ is the future inextendable geodesic with $\gamma_v(0) = a$ and $\dot{\gamma}_v(0) = v$. Now we give the definitions of $\mathcal{I}^+(a)$ and $\mathcal{I}^-(a)$ as following.

Definition 5. Let $(N_1^{n+1}, \langle\cdot\rangle)$ be a spacetime of signature $(1, n)$, and $a \in N_1^{n+1}$. Define $\tilde{\mathcal{I}}^+(a) = \{tv : v \in T_{-1}N \mid_a \text{ and } 0 < t < s_a(v)\}$ and $\mathcal{I}^+(a) = \exp_a(\text{int}(\tilde{\mathcal{I}}^+(a)))$. The set $\mathcal{I}^-(a)$ can be defined dually.

We have the following result for the smoothness of the Lorentzian distance function(see e.g., [5]).

Lemma 6. *If $\mathcal{I}^+(a) \neq \emptyset$ (resp. $\mathcal{I}^-(a) \neq \emptyset$), then the function $d(a, \cdot)$ (resp. $d(\cdot, a)$) is smooth on $\mathcal{I}^+(a)$ (resp. $\mathcal{I}^-(a)$).*

Now let $\sigma : [0, r] \to N_1^{n+1}$ be an unit-speed timelike geodesic. The index form I_σ on σ is defined by [6]

$$I_\sigma(V, W) = -\int_0^r (\langle D_T V, D_T W\rangle - \langle R(V, T)T, W\rangle)\,\mathrm{d}t, \tag{3}$$

where $T = \dot\sigma$ is the tangent vector of σ, D the Levi-Civita connection of N_1^{n+1}, R the curvature tensor of N_1^{n+1} and V, W are the vector fields along σ and perpendicular to σ, respectively. A vector field J along σ is called the Jacobi field if it satisfies the following equation:

$$D_T D_T J = R(T, J)T. \tag{4}$$

We have

Lemma 7. (Maximality of Jacobi fields [4]) *Let N_1^{n+1} be a spacetime of signature $(1, n)$, and $\sigma : [0, r] \to N_1^{n+1}$ be an unit-speed timelike geodesic such that there are no conjugate points of $\sigma(0)$ along σ. Let J be a Jacobi field on σ and X be a vector field on σ such that $X(0) = J(0)$, $X(r) = J(r)$ and J, X are orthogonal to σ. Then $I_\sigma(X, X) \le I_\sigma(J, J)$.*

Lemma 8. *Let N_1^{n+1} be a spacetime of signature $(1, n)$ with $K(\pi) \ge c$ for any mixed 2-plane π of signature $(1, 1)$ in N_1^{n+1}, and $\sigma : [0, r] \to N_1^{n+1}$ an unit timelike geodesic (when $c < 0$, we assume that $r < \pi/(2\sqrt{-c})$). Let J be a Jacobi field on σ such that $J(0) = 0$ and $J(r) \perp T$. Then*

$$I_\sigma(J, J) \le -\alpha(c, r)\mid J(r)\mid^2 .$$

Proof. We prove the lemma for $c < 0$, other cases can be verified similarly. In this case, let $\tilde\sigma : [0, r] \to H_1^{n+1}(c)$ be a unit-speed timelike geodesic in $H_1^{n+1}(c)$. Since $r < \pi/(2\sqrt{-c})$, there exist no conjugate points of $\tilde\sigma(0)$ along $\tilde\sigma$. Choose the Lorentzian frame $\tilde e_1(t), \cdots, \tilde e_{n+1}(t)$ of $H_1^{n+1}(c)$ along $\tilde\sigma$ such that $\tilde e_1(t), \cdots, \tilde e_n(t)$ are parallel along $\tilde\sigma$ and $\tilde e_{n+1} = \tilde T = \dot{\tilde\sigma}$, the tangent vector field of $\tilde\sigma$. Let

$$\tilde J(t) = \mid J(r)\mid \frac{\sin(\sqrt{-c}t)}{\sin(\sqrt{-c}r)}\tilde e_1(t)$$

be a Jacobi field on $\tilde\sigma$. Similarly, we choose the Lorentzian frame $e_1(t), \cdots, e_{n+1}(t)$ of N_1^{n+1} along σ such that $e_1(t), \cdots, e_n(t)$ are parallel along σ, $e_{n+1} = T$ and $J(r) = \mid J(r)\mid e_1(r)$. Write $J(t) = \sum_{i=1}^n h_i(t)e_i(t)$. Let $\tilde X = \sum_{i=1}^n h_i(t)\tilde e_i(t)$ be a vector field along $\tilde\sigma$. Then by Lemma 7 and the assumptions of Lemma 8 we get

$$\begin{aligned} I_\sigma(J, J) &= -\int_0^r (\langle D_T J, D_T J\rangle - \langle R(J, T)T, J\rangle)\,\mathrm{d}t \le -\int_0^r \left(\sum_{i=1}^n \left(\dot h_i^2 + c h_i^2\right)\right)\mathrm{d}t \\ &= -\int_0^r \left(\langle \tilde D_{\tilde T}\tilde X, \tilde D_{\tilde T}\tilde X\rangle - \langle \tilde R(\tilde X, \tilde T)\tilde T, \tilde X\rangle\right)\mathrm{d}t = I_{\tilde\sigma}(\tilde X, \tilde X) \le I_{\tilde\sigma}(\tilde J, \tilde J) \\ &= -\alpha(c, r)\mid J(r)\mid^2, \end{aligned}$$

where $\tilde{D}$ and $\tilde{R}$ are the Levi-Civita connection and the curvature tensor of $H_1^{n+1}(c)$, respectively. So the lemma is proved. □

3. The proof of theorems

We shall complete the proof of Theorems 2 and 4 in this section. Let $\psi : M^n \to N_1^{n+1}$ be a spacelike hypersurface in Lorentzian $(n+1)$-manifold, and ∇ be the Levi-Civita connection on M. Then the Gauss and Weingarten formulas for M in N_1^{n+1} are given respectively by

$$D_X Y = \nabla_X Y - \langle AX, Y\rangle \xi \tag{5}$$

and

$$A(X) = -D_X \xi \tag{6}$$

for all tangent vector fields $X, Y \in \chi(M)$. Here ξ is the local unit timelike normal vector field along $\psi(M)$, and $A : \chi(M) \to \chi(M)$ stands for the shape operator of M in N_1^{n+1} associated to ξ.

Associated to the shape operator of M there are n algebraic invariants, which are the elementary symmetric functions σ_j of its principal curvatures $\lambda_1, \cdots, \lambda_n$, given by

$$\sigma_j(\lambda_1, \ldots, \lambda_n) = \sum_{i_1<\ldots<i_j} \lambda_{i_1} \ldots \lambda_{i_j}, \quad 1 \le j \le n.$$

The jth mean curvature H_j of the spacelike hypersurfaces is then defined by

$$\binom{n}{j} H_j = \sigma_j(\lambda_1, \cdots, \lambda_n). \tag{7}$$

We note that the jth mean curvature H_j is intrinsic for even j and is extrinsic for odd j.

Now we are ready to prove Theorems 2 and 4.

Proof of Theorem 2. Without loss of generality we assume that $\psi(M) \subset \mathcal{I}^+(a)$, and $\tilde{\rho} = d(a, \cdot) : N_1^{n+1} \to R$ be the Lorentzian distance function on N_1^{n+1} with respect to a. According to Lemma 6 we see that the function $\rho = \tilde{\rho}\circ\psi : M \to R$ is smooth on M, and let us first compute the gradient and Hessian of function ρ. For convenience we identify $\psi(M)$ with M. For $b \in \psi(M)$, let $\sigma : [0, \rho(b)] \to N_1^{n+1}$ be the maximal timelike geodesic from a to b, and X be the unit tangent vector of $\psi(M)$ at b. By parallel translation we get a vector field $X(t)$ along σ. Let $\zeta : [-\varepsilon, \varepsilon] \to M$ be the geodesic on M satisfying $\zeta'(0) = X$. For $u \in [-\varepsilon, \varepsilon]$, let $\sigma_u : [0, \rho(b)] \to N_1^{n+1}$ be the maximal timelike geodesic from a to $\zeta(u)$. Then $\Gamma(t, u) = \sigma_u(t) : [0, \rho(b)] \times [-\varepsilon, \varepsilon] \to N_1^{n+1}$ be a family of one-parameter geodesics with $\sigma_0(t) = \sigma(t)$, $\sigma_u(0) = a$, $\sigma_u(\rho(b)) = \zeta(u)$ and $L(\sigma_u) = \rho(\zeta(u))$. Let $J(t)$ be the Jacobi field along σ induced by Γ, then $J(0) = 0$, $J(\rho(b)) = X$. It follows from the first and second variations of arc length (see e.g., [6]) that

$$\nabla\rho(X) = \frac{\mathrm{d}}{\mathrm{d}u} L(\sigma_u) \mid_{u=0} = -\langle X(b), T(b)\rangle, \tag{8}$$

$$\nabla^2\rho(X, X) = \frac{\mathrm{d}^2}{\mathrm{d}u^2} L(\sigma_u) \mid_{u=0} = \langle AX, X\rangle \langle \xi(b), T(b)\rangle + I_\sigma(J^\perp, J^\perp), \tag{9}$$

where $J^\perp = J + \langle J, T\rangle T$. Since $\inf_M d(a, \psi) = r$, we have $\rho \ge r$ or $-\rho \le -r$. By the generalized maximum principle [7] we see that for any $\varepsilon > 0$ there exists a point $b_\varepsilon \in M$ such that

$$\mid \nabla\rho(b_\varepsilon) \mid < \varepsilon, \tag{10}$$

$$\nabla^2\rho(b_\varepsilon)(X, X) > -\varepsilon, \tag{11}$$

for all unit vector $X \in T_{b_\varepsilon}M$, and

$$r \le \rho(b_\varepsilon) \le r + \varepsilon. \tag{12}$$

Let $e_1, \dots, e_n$ be the orthonormal frame for $T_{b_\varepsilon}M$, and write $T(b_\varepsilon) = \sum_{i=1}^n \langle T, e_i\rangle e_i - \langle T, \xi\rangle\xi$, then from (8) and (10) we get

$$-1 = -\langle T, \xi\rangle^2 + \sum_{i=1}^n \langle T, e_i\rangle^2 < -\langle T, \xi\rangle^2 + \varepsilon^2,$$

and consequently,

$$\langle T, \xi\rangle^2(b_\varepsilon) < 1 + \varepsilon^2. \tag{13}$$

Combining (9),(11)–(13) and Lemma 8 we see that

$$\begin{aligned}-\varepsilon &< \langle AX, X\rangle\langle\xi, T\rangle(b_\varepsilon) - \alpha(c, \rho(b_\varepsilon)) \mid J^\perp(b_\varepsilon) \mid^2 \le \langle AX, X\rangle\langle\xi, T\rangle(b_\varepsilon) \\ &\quad - \alpha(c, r+\varepsilon)(1 + \langle X, T\rangle^2) \le \langle AX, X\rangle\langle\xi, T\rangle(b_\varepsilon) - \alpha(c, r+\varepsilon).\end{aligned}$$

Therefore,

$$\alpha(c, r+\varepsilon) - \varepsilon < \langle AX, X\rangle\langle\xi, T\rangle(b_\varepsilon) \le \mid \langle AX, X\rangle \mid \sqrt{1+\varepsilon^2}. \tag{14}$$

It is easy to know from (14) that the principal curvatures $\lambda_1, \dots, \lambda_n$ of M in N_1^{n+1} at the point b_ε have the same sign when ε is sufficiently small, and that

$$\mid \lambda_i \mid \ge \frac{\alpha(c, r+\varepsilon) - \varepsilon}{\sqrt{1+\varepsilon^2}} \to \alpha(c, r) \quad (\varepsilon \to 0). \tag{15}$$

Thus, from the definition (7) of the jth mean curvature H_j we have

$$\sup_M \mid H_j \mid \ge \alpha(c, r)^j.$$

This finishes the proof of Theorem 2. □

Proof of Theorem 4. We prove the theorem for $c > 0$, other cases can be shown similarly.

Let $\psi : M^n \to S_p^{n+p}(c)$ be a complete spacelike hypersurface in de Sitter space of constant curvature c. Recall that $S_p^{n+p}(c) = \{x \in R_p^{n+p+1} : \langle x, x\rangle = \frac{1}{c}\} \subset R_p^{n+p+1}$. For $a \in S_p^{n+p}(c)$, let $f = \langle a, \cdot\rangle : M \to R$. Then it is easy to know that

$$\nabla f(X) = \langle a, X\rangle \tag{16}$$

for any unit tangent vector of M, and

$$\Delta f = n\langle a, \mathbf{H}\rangle - ncf, \tag{17}$$

where Δ is the Laplacian operator of M. Since $\inf_M\langle a, \psi\rangle = \frac{1}{c}\cosh(\sqrt{c}r)$, by the generalized maximal principle [7] we see that for any $\varepsilon > 0$, there exists a point $b_\varepsilon \in M$ such that at b_ε we have

$$\mid \nabla f \mid < \varepsilon, \quad \Delta f > -\varepsilon, \quad \frac{1}{c}\cosh(\sqrt{c}r) + \varepsilon \ge f \ge \frac{1}{c}\cosh(\sqrt{c}r). \tag{18}$$

Let $e_1, \ldots, e_{n+p}$ be the semi-Riemannian frame of $S_p^{n+p}(c)$ at b_ε such that $e_1, \ldots, e_n$ are tangent to M and $\mathbf{H} = |\mathbf{H}| e_{n+1}$. Since $a \in S_p^{n+p}(c)$, it follows from (16) and (18) that

$$\frac{1}{c} = \langle a, a \rangle = \sum_{i=1}^{n} \langle a, e_i \rangle^2 + \langle a, \sqrt{c}\psi \rangle^2 - \sum_{\alpha=n+1}^{n+p} \langle a, e_\alpha \rangle^2 < \varepsilon^2 + c\left(\frac{1}{c}\cosh(\sqrt{c}r) + \varepsilon\right)^2 - \langle a, e_{n+1} \rangle^2,$$

and consequently,

$$\langle a, e_{n+1} \rangle^2 < \varepsilon^2 + c\left(\frac{1}{c}\cosh(\sqrt{c}r) + \varepsilon\right)^2 - \frac{1}{c}. \tag{19}$$

Combining (17)–(19) we get

$$-\frac{\varepsilon}{n} < \frac{1}{n}\triangle f < |\mathbf{H}| \sqrt{\varepsilon^2 + c\left(\frac{1}{c}\cosh(\sqrt{c}r) + \varepsilon\right)^2 - \frac{1}{c}} - \cosh(\sqrt{c}r). \tag{20}$$

Finally, we have

$$\sup_M |\mathbf{H}| > \frac{\cosh(\sqrt{c}r) - \frac{\varepsilon}{n}}{\sqrt{\varepsilon^2 + c\left(\frac{1}{c}\cosh(\sqrt{c}r) + \varepsilon\right)^2 - \frac{1}{c}}} \to \sqrt{c}\coth(\sqrt{c}r) \quad (\varepsilon \to 0),$$

and the theorem is proved. □

Acknowledgements

This work was accomplished during my post-doctor period in Institute of Mathematics, Fudan University. I would like to express my sincere thanks to Professor Y.L. Xin for many valuable discussions on this work. Also, I would like to thank the referee whose valuable suggestions make this paper more perfect.

References

[1] J.A. Aledo, L.J. Alías, On the curvatures of bounded complete spacelike hypersurfaces in the Lorentz–Minkowski space, Manuscripta Math. 101 (2000) 401–413.

[2] J.A. Aledo, L.J. Alías, On the curvature of complete spacelike hypersurfaces in de Sitter space, Geom. Dedicata 80 (2000) 51–58.

[3] J.A. Aledo, J.A. Gálvez, A. Romero, Some estimates for the curvatures of complete spacelike hypersurfaces in generalized Robertson-Walker spacetimes, J. Geom. Phys. 52 (2004) 469–479.

[4] J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, Marcel Dekker Inc., New York, 1996.

[5] F. Erkekoglu, E. Garcia-Rio, D.N. Kupeli, On level sets of Lorentzian distance function, Gen. Rel. Grav. 35 (2003) 1597–1615.

[6] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[7] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Jpn. 19 (1967) 205–214.